

Emergence of firms in $(d + 1)$ -dimensional work space

G Weisbuch¹, D Stauffer², D Mangalagiu³, R Ben-Av⁴, S Solomon
Racah Institute of Physics, Hebrew University, IL-91904 Jerusalem, Israel

¹ Visiting from Laboratoire de Physique Statistique, ¹
Ecole Normale Supérieure, F-75231 Paris, France

gerard.weisbuch@gmail.com

² Visiting from Institute for Theoretical Physics,
Cologne University, D-50923 Köln, Euroland

³ Management and Strategy Department, Reims Management School,
59, rue Pierre Taittinger, F-51061 Reims Cedex, France

⁴ Department of Software Engineering, Jerusalem College of Engineering
(JCE), Israel

1 Introduction

Standard micro-economics concentrate on the description of markets but is seldom interested in production. Several economists discussed the concept of a firm, as opposed to an open labour market where entrepreneurs would recruit workers on the occasion of each business opportunity. Coase [1] is one of them, who explains the existence of firms as institution because they reduce the transaction costs with respect to an open labour market.

Whatever the rationale proposed by economists to account for the existence of firms, their perspective is based on efficiency and cost analysis. Little attention is paid to the dynamics of emergence and evolution of firms.

The aim of the present manuscript is to check the global dynamical properties of a very simple model based on bounded rationality and reinforcement learning. Workers and managers are localised on a lattice and they choose collaborators on the basis of the success of previous work relations. The choice algorithm is largely inspired from the observation and modeling of long term customer/sellers relationships observed on perishable goods markets discussed in Weisbuch et al[2] and Nadal et al[3].

The model presented here is in no way an alternative to Coase. We describe the build-up of long term relationships which do reduce transaction

¹Laboratoire associé au CNRS (UMR 8550), à l'ENS et aux Univ. Paris 6 et Paris 7

costs, and we deduce the dynamical properties of networks built from our simple assumptions.

2 The model

2.1 The model

Let us imagine a production network of workers: we use the simplest structure of a lattice: at each node is localised a "worker" with a given production capacity of 1. Business opportunities of size Q randomly strike "entrepreneur" sites at the surface of the lattice.

The work load received by the entrepreneur is too large to be carried out by her: she then distributes it randomly to her neighbours upstream; let us say that these neighbours are her nearest neighbours upstream. We postulate two mechanisms here: a probabilistic choice process according to preferences to different neighbours, and the upgrading of preferences by as a function of previous gains. The probability of choosing neighbour j is given by the logit function:

$$p_j = \exp(\beta J_j) / \sum_{k=1}^{nb} \exp(\beta J_k) \quad (1)$$

where the sum extends to all neighbours of the node. The preferences J_j are updated at each time step according to:

$$J_j(t) = (1 - \gamma)J_j(t - 1) + q_j(t) \quad (2)$$

where $q_j(t)$ is the work load attributed to node j .

One time step corresponds to the distribution of work loads across the set of collaborators of the entrepreneur who received the work load.

A series of work loads strike the entrepreneur at successive time steps. We want to characterise the asymptotic structure generated by a large number of work loads presented in succession to the entrepreneur.

2.2 The algorithm

Workers are placed on a $(d + 1)$ -dimensional hyper-cubic lattice of height L_z . Each hyper-plane $line = 1, 2, \dots, L_z$ is a lattice of linear dimension L with L^d sites and periodic (helical) boundary conditions. A workload Q is

distributed from the top level (hyper-plane) $line = 1$ upstream, in steps from level $line$ to level $line + 1$, until Q different workers (sites) i each have a local workload $q_i = 1$. All local workloads q_i are integers $1, 2, \dots, Q$.

One iteration corresponds to the downward distribution of one workload and proceeds as follows: Initially all sites have workload zero. A new workload arrives at the central site of the top level. Thereafter, each site i on level $line$ having a local workload $q_i > 1$ distributes the surplus $q_i - 1$ to its $n_{nb} = 2d + 1$ nearest and next-nearest neighbours j on the lower level $line + 1$, in unit packets $q_i \rightarrow q_i - 1$. For this purpose it selects, again and again, randomly one such neighbour j and transfers to it with probability p , given by equation (1) one unit of workload, increasing by one unit the preference J_{ij} storing the history of work relations. After site i has distributed its workload in this way to the lower level of hierarchy and has only a remaining unit workload, the algorithm moves to the next site having a local workload bigger than unity. The whole iteration stops when the lowest level $line = L_z$ is reached or when no site has a local workload above unity. Then all workloads q_i are set back to zero, all stored preferences J are diminished by a factor $1 - \gamma$, and a new iteration starts, influenced by the past history stored in the preferences J_{ij} .

3 Simulation results

3.1 Equilibration

Fig.1 shows that for the chosen parameters some stationary equilibrium is obtained between the increase of sum of all J_{ij} , called the flow, due to new work, and the decrease of the flow due to the forgetting parameter. The depth of the load pattern in the lattice also increases and finally reach saturation as the flow. These dynamics are similar in lower (1+1) and higher (1+4) dimensions.

3.2 Snakes and blobs

According to β , γ and load values, after many iterations two dynamical regimes are observed: a quasi-deterministic regime such that only one link out of $2d + 1$ is systematically chosen resulting in a "snake" portion of the work pattern, and a random regime where all 3 links are used, resulting in a

”blob” portion of the work pattern. The interface between the two regimes corresponds to

$$\beta(q(z) - 1)/\gamma = \text{Constant} \quad (3)$$

where $q(z) - 1$ is the work load distributed by a node of charge $q(z)$ at depth z . Because the work load to be distributed, $q(z) - 1$ decreases with increasing depth z , there is a given depth where the interface between the deterministic regime and the random regime is located.

Figure 2 displays workloads obtained in a (1+1)-dimensional lattice. On this figure the snakes extends from the initial load of 20 to the load of 7 followed by a small blob of height 3. Parameters for this simulation were $\gamma = 0.3, \beta = 0.3$.

A mean field theory, proposed in a different context by Weisbuch et al[2] and Nadal et al[3], predicts a transition between the head and tails regime at a depth z such that:

$$\frac{\beta * (q(z) - 1)}{\gamma} = 2d + 1 \quad (4)$$

For larger values of $q(z)$, all the work load is transferred to a single neighbour with a preference coefficient of $(q(z) - 1)/\gamma$, and all other coefficients are 0. For lower values of $q(z)$, all preference coefficients are small, with possible fluctuations around the interface. These predictions are verified in figure 3 computed for simulations in 1+1, 1+2 and 1+3 dimensions.

Figure 4 is a more systematic test in 1+1 dimension of this dynamics. We here plotted in the mean square distance of the positions of the workers in each hyper-plane = line from the position of the highest worker concentration (more precisely: from the center of mass of their distribution). In the tail this squared width is exactly zero (left part), while in the blob (right part for each curve) it has a peak. The peak position shifts from small depths (close top plane, no snake) to large depths (close to bottom plane at 60, long snake), when beta increases. Similar plots of the snake lengths were obtained for $d = 2$ and 3 (not shown); also one test for $d = 4, L = 29$ displays an interface between zero and positive width.

We plot in Fig.5 the average (over ten samples of 10,000 iterations each) of the position of the lowest hyper-plane touched by the work distribution process as a function of β . Although the statistics obtained are a clear indication that the depth of the system follows the same trend from 1+1 to 1+3 dimensions they are not directly interpreted. The measured depth is

in fact the sum of the length of the snake part plus the height of the blob part. Both parts vary with β . The snake length is obtained from equation (3) since $q(z)$ is simply $Q - z$. But the blob height depends upon the charge at the interface and we don't have any simple analytic expression for it.

Figure 6 shows the total number N_t of sites which were involved in at least one of the 10,000 iterations, i.e. the total work force with long-time employment, and also at the fluctuations N_f in the work force from one iteration to the next. (Thus N_f is the number of sites which are used at iteration t and not at time $t - 1$, or the other way around.) We see that for large β the fluctuations are diminished (the same snake tail again and again passes on the work), but this decrease is accompanied by an increase in N_t , an effect which helps the labour market but not the company.

In the above version, also the managers who distributed work to their lower neighbours took over one work unit each for themselves. If instead they give on all the workload given to them (provided it was at least two units), then each iteration requires not only Q people as above, but Q workers plus a fluctuating number of managers. Moreover, if the snake hits the bottom line at depth $L_z = Q$, part of the work is never finished. This is hardly an efficient way to run a business, but Figure 7 and 8 show a much sharper transition, from a localised cluster at low β to delocalized snake tails at high β .

4 Conclusions

The simple reinforcement learning presented here does end-up in a metastable path in the worker space represented here by the snake + blob picture, which we interpret as a firm. On the other hand we would rather imagine firms as hierarchical structures such as trees [4, 5]. Because of the blob-snake sharp transition as a function of z , we never observe a well balanced tree with a selection at each node of several preferred collaborators, but rather either a nearly complete preference for one neighbour or roughly equal preference for all.

In conclusion, the present model explains the metastability of employment relations in the firm, but something has to be added to it to explain the more efficient workload repartition observed in real firms.

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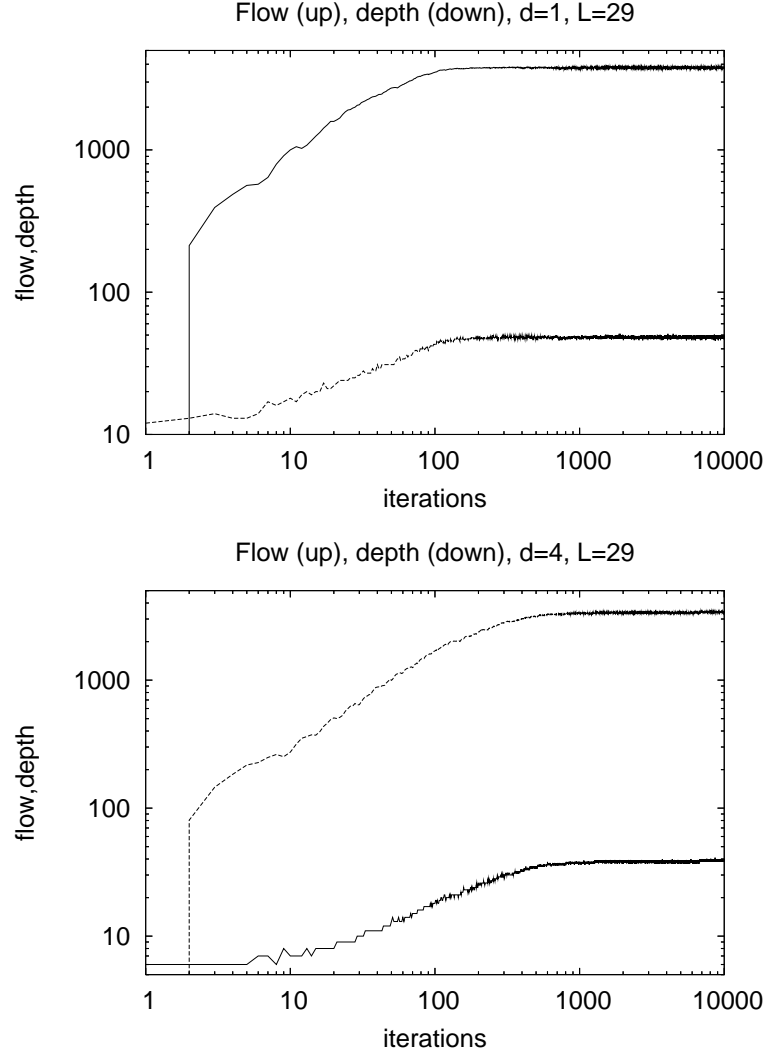


Figure 1: Top: Equilibration of the sum of all preference coefficients (top curve), and depth at $\beta = 0.1$, $\gamma = 0.3$, $1 + 1\text{dimensions}$, $Q = 60$, averaged over 10,000 iterations. Bottom: Same parameters except for $d = 4$.

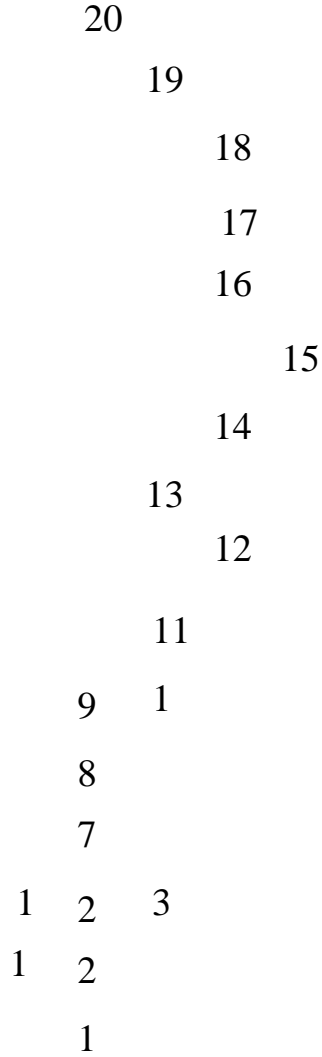


Figure 2: One instance of the work load repartition from an initial load of 20 at the top site until the lower line. The load is initially distributed with a strong preference for one neighbour out of three and is then more uniformly from load 7. $\beta = 0.3, \gamma = 0.3, Q = 20$.

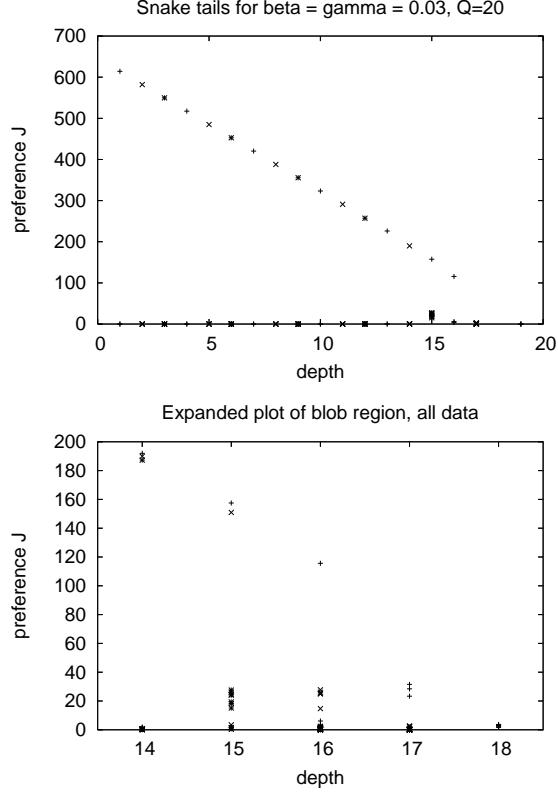


Figure 3: Evolution in $1+d$ dimensions of the preference coefficient with distance from the "surface". For smaller depths, in the tail region, the preference coefficients are either strong $((q(z)-1)/\gamma)$ and independent of the dimension d , or zero. A transition is observed around resp. charges of 5, 4 and 3 rather than for $2d+1 = 7, 5$ and 3 resp. as predicted by the mean field theory (equation (3)). Top part: Overall picture emphasising the snake. Bottom part: Enlargement of blob region. $Q = 20$, $\beta = 0.03$, $\gamma = 0.03$, $d = 1(+), 2(x), 3(*)$.

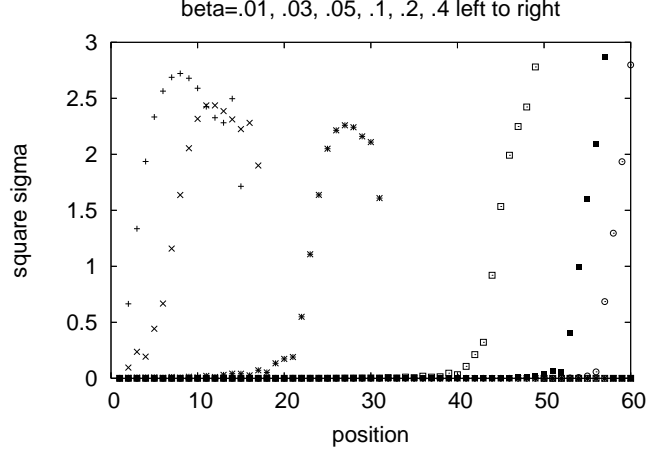


Figure 4: Variation with vertical position of the square width of the working region within a horizontal hyper-plane (= line), averaged over the last 5,000 of 10,000 iterations. $\gamma = 0.3$, $d = 1$, $Q = 60$ Similar results were obtained for $d = 2$ and 3.

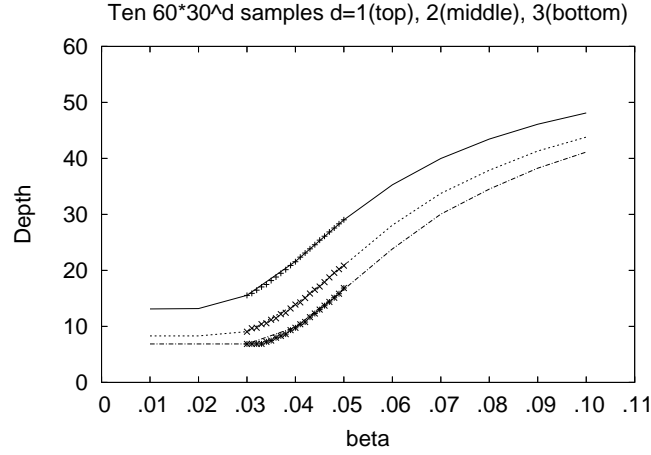


Figure 5: Variation with β of the vertical position of the lowest working plane for several dimensions (1+1,1+2,1+3). Enlargement to $L = 300$ or $L_z = 600$ gives no significant changes.

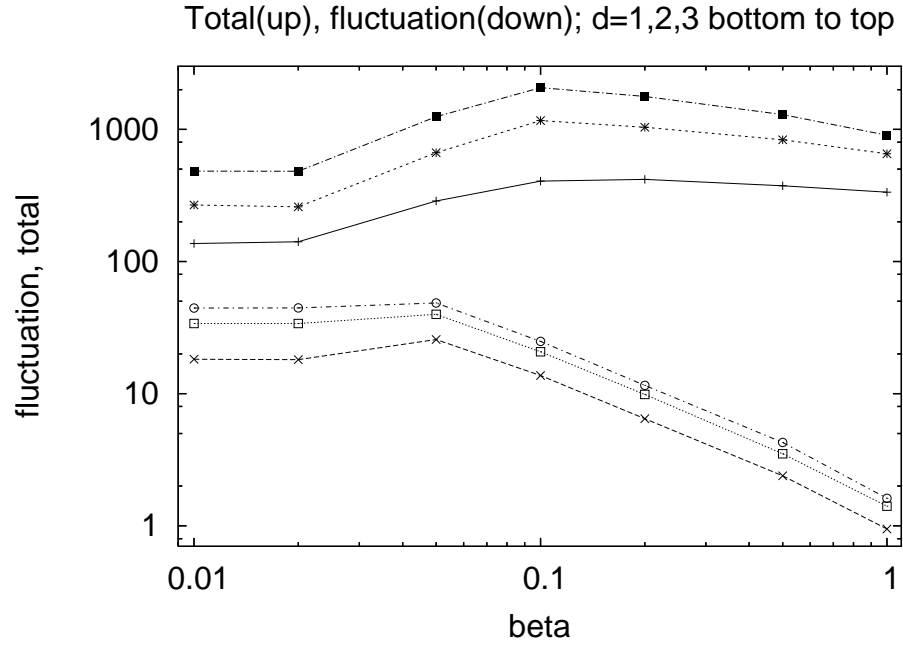


Figure 6: Variation with β of the total number of different people who worked during at least one of the 10,000 iteration (three top curves), and of the fluctuations in the work force (three bottom curves); $L = 30$, $L_z = 60$, one sample. Data are averages over the last 1000 iterations.

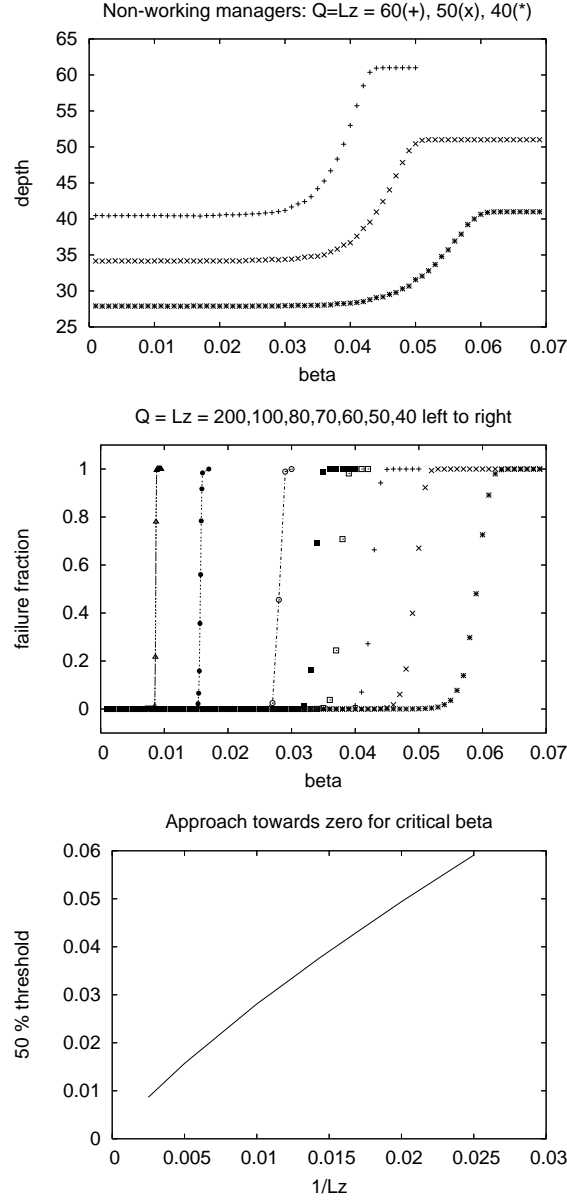


Figure 7: Results for the non-working managers model; ten samples for 10,000 iterations each, $d = 1$, $L = 30$, $Q = L_z$ increases from right to left. Top: Average position of lowest working plane. Middle: Average fraction of failures where process hits the lattice bottom at L_z . Bottom: β value according to middle part where the failure fraction reaches 50 percent. $\gamma = 0.3$.

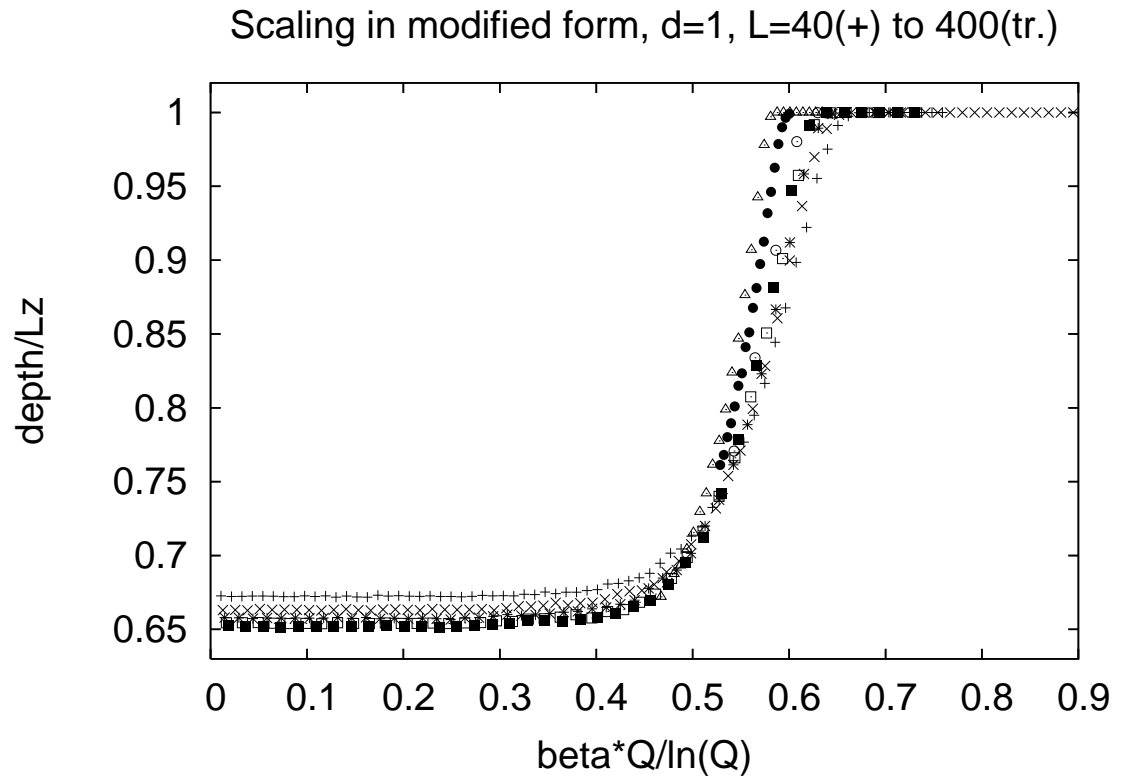


Figure 8: Scaling plot of figure 7, top, with approximate collapse for different $Q = Lz$ between 40 to 400.